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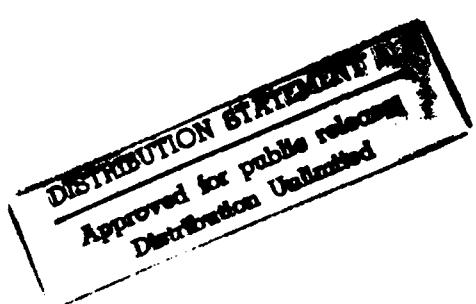


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**Optimal Shear Correction Factors
in
Hierarchical Plate Modelling**

by

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and
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Optimal Shear Correction Factors in Hierarchical Plate Modelling

by

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Abstract

This paper addresses the formulation of hierarchical plate models as an optimal (in a clearly defined sense) numerical method which can be easily implemented and is available in the code MSC/PROBE. Special emphasis is placed on the optimal selection of the shear correction factors. It is shown that different measures of accuracy lead to different optimal choices of these factors. The main tool in the analysis is the Fourier transformation.

1. Introduction

The problem of plate modelling is a classical problem of mechanics. The principles underlying the derivation of plate models can be divided into 3 broad groups.

- a) *Physical derivation.* Here assumptions of a geometrical nature are made a priori in conjunction with additional assumptions involving certain stress components. A typical example is the Kirchhoff hypothesis [26] which leads to the well known biharmonic equation for the normal deflection of the plate. This approach has been generalized in many papers and we refer here especially to [1, 2, 10, 17, 23, 24, 25, 30, 31, 34, 35, 37, 39, 40, 41, 44, 46, 49] and the references therein. The idea of a shear correction factor was first introduced in the papers of this group.
- b) *Asymptotic Analysis.* Here plate models are justified as the leading terms of an asymptotic expansion of the three-dimensional solution with respect to the plate-thickness. We refer here for example to [19], [20] and to [34], pp. 585-588, where a historical account for this approach is given. Here the asymptotic methods were mostly applied to the plate problem formulated as a differential equation rather than to its variational form. Asymptotic methods, where a mixed variational principle was used, appeared only recently [11]. See also [7], [33] for related ideas.
- c) *Numerical and Hierarchical Approach.* Here we understand the modelling as approximate solution of the three dimensional problem (which is solved for example by the finite element method). It is essential to use an approach which can be easily implemented (and which avoids in particular higher order equations) in the framework of the $h-p$ version of the finite element method and which has optimal properties yielding the most accurate results when compared with the exact solution of the three dimensional problem. It has to be emphasized that various plate models can significantly differ in the area where boundary layers and singularities dominate the solution behaviour. Hence a flexible hierarchy is essential (and has been implemented in the code MSC/PROBE). A posteriori error estimators and adaptive approaches for the model selection within the hierarchy are yet to be developed.

In this paper we concentrate on the selection of the shear correction factor which leads to the highest accuracy of the solution. We restrict ourselves to the case of an infinite plate made of a homogeneous, isotropic material and use the Fourier transform technique in our analysis. The problem of anisotropic and laminated plates will be discussed elsewhere. In the second section we define the three dimensional problem and the hierarchy of the models. The third section addresses the Fourier transform of the solution for the three dimensional formulation as well as for the plate model problem. In section four we analyze the errors of various models and give the optimal shear factors. In section five we present numerical examples and in section six we sum up our basic conclusions.

2. Hierarchical Plate Models

2.1. The Plate Problem

The *plate problem* is the boundary value problem of three dimensional, linear elasticity for an isotropic material on the domain

$$\Omega^p = \{\mathbf{x} = (x_1, x_2, x_3) \mid (x_1, x_2) \in \omega, |x_3| \leq d\} = \omega \times (-d, d) \quad (2.1)$$

where $\omega \subset R^2$ is a bounded domain with a piecewise smooth boundary. Here ω is referred to as the midsurface and $2d$ as the thickness of the plate.

As usually, we denote by $u = \{u_i\}$, $i = 1, 2, 3$ the displacement vector. By $\sigma = \{\sigma_{ij}\}$, $i, j = 1, 2, 3$ and $e = \{e_{ij}\}$, $i, j = 1, 2, 3$ we denote the stress and strain tensor, respectively, with $e_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$, $i, j = 1, 2, 3$. Hooke's law relating the stress and strain can then be written in the form

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{bmatrix} = \begin{bmatrix} \lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda + 2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \begin{bmatrix} e_{11} \\ e_{22} \\ e_{33} \\ 2e_{12} \\ 2e_{13} \\ 2e_{23} \end{bmatrix} \quad (2.2)$$

or

$$\sigma = Ae, \quad A = \{a_{ij}\}, \quad a_{ij} = a_{ji}, \quad i, j = 1, \dots, 6. \quad (2.3)$$

Here A is the material matrix, λ, μ are the Lamé constants, and

$$\lambda = \frac{\nu E}{(1 + \nu)(1 - 2\nu)} = a_{12} = a_{13} = a_{23}$$

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$$\begin{aligned}
\lambda + 2\mu &= \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} = a_{11} = a_{22} = a_{33}, \\
\mu &= \frac{E}{2(1+\nu)} = G = a_{44} = a_{55} = a_{66}, \\
\text{and } a_{ij} &= 0, 4 \leq i \leq 6, \quad j = 1, \dots, 6, \quad j \neq i.
\end{aligned} \tag{2.4}$$

Here E is Young's modulus and ν is the Poisson ratio.

We will focus on the classical plate bending problem where the normal tractions are prescribed on the upper and lower plate surface. For $(x_1, x_2) \in \omega$ we have

$$\begin{aligned}
\sigma_{33}(x_1, x_2, \pm d) &= \pm \frac{1}{2}q(x_1, x_2) \\
\sigma_{\alpha 3}(x_1, x_2, \pm d) &= 0, \quad \alpha = 1, 2
\end{aligned} \tag{2.5}$$

where $q(x_1, x_2) \in L_2(\omega)$. The solution u is the minimizer of the total plate energy $\mathcal{G}_p^A(u)$

$$\mathcal{G}_p^A(u) = \frac{1}{2}\mathcal{E}_p^A(u) - \int_{\omega} \frac{1}{2}q(x)(u_3(x_1, x_2, d) + u_3(x_1, x_2, -d))dx_1dx_2, \tag{2.6}$$

where

$$\mathcal{E}_p^A(u) = \int_{\Omega^p} e^T A e dx = \|u\|_E^2$$

is the strain energy, over a suitable space $\mathcal{H}(\Omega^p) \subset (H^1(\Omega^p))^3$ where $\mathcal{H}(\Omega^p)$ characterizes the boundary conditions on the lateral plate sides and $H^1(\Omega)$ denotes the usual Sobolev space.

By

$$\mathcal{N}^p = \{u \mid u = (a_1, a_2, a_3) + (b_1, b_2, b_3) \times (x_1, x_2, x_3), \quad a_i, b_i \in R \quad i = 1, 2, 3\} \tag{2.7}$$

we denote the set of rigid body motions. Now, if $\mathcal{N}_{\mathcal{H}(\Omega^p)}^p = \mathcal{N}^p \cap \mathcal{H}(\Omega^p) = \emptyset$ and $q \in L_2(\omega)$, the quadratic functional $\mathcal{G}_p^A(u)$ has a unique minimizer. This follows from Korn's inequality (for an elementary proof of which we refer to [27]) and the standard use of the Lax-Milgram lemma.

It is obvious that due to the symmetry of the problem, we can constrain the space of solutions to be of the form

$$\begin{aligned}
u_1(x_1, x_2, x_3) &= -u_1(x_1, x_2, -x_3) \\
u_2(x_1, x_2, x_3) &= -u_2(x_1, x_2, -x_3) \\
u_3(x_1, x_2, x_3) &= u_3(x_1, x_2, -x_3).
\end{aligned} \tag{2.8}$$

Relation (2.8) constraints the space $\mathcal{H}(\Omega^p)$ accordingly and will be assumed everywhere in what follows. In the case that $\mathcal{N}^p \cap \mathcal{H}(\Omega^p) = \mathcal{N}_{\mathcal{H}(\Omega^p)}^p \neq \emptyset$ we have to assume in addition that

$$\int_{\omega} (0, 0, q) \cdot \mathcal{R}^T dx = 0 \quad \forall \mathcal{R} \in \mathcal{N}_{\mathcal{H}(\Omega^p)}^p, \tag{2.9}$$

then the minimizer of (2.6) exists and is unique modulo $\mathcal{N}_{\mathcal{H}(\Omega^p)}^p$.

For $\omega = R^2$ and $\Omega_\infty^p := \omega \times (-d, d)$ the situation is slightly more complicated. In this case we will assume that $q(x_1, x_2) \in L_2(R^2)$, has compact support and satisfies (2.9) for any $\mathcal{R} \in \mathcal{N}^p$. Then the quadratic functional \mathcal{G}_p^A has a minimizer over $\mathcal{H}(\Omega_\infty^p)$ where

$$\mathcal{H}(\Omega_\infty^p) = \{u_i \in H_{\text{loc}}^1(\Omega_\infty^p), i = 1, 2, 3 \mid \int_{\Omega_\infty^p} e^T A e \, dx < \infty\}. \quad (2.10)$$

The space $\mathcal{H}(\Omega_\infty^p)$ is not a subspace of $(H^1(\Omega_\infty^p))^3$ because the displacement components $u_i \in \mathcal{H}(\Omega_\infty^p)$ are in general not square integrable. For more about the analysis of the characteristics of spaces of the type $\mathcal{H}(\Omega_\infty^p)$ and related questions we refer to [27]. The minimizer of \mathcal{G}_A^p is unique modulo functions in \mathcal{N}^p and once more we will constrain the space $\mathcal{H}(\Omega_\infty^p)$ by (2.8).

We will also address the $\alpha = (\alpha_1, \alpha_2)$ -periodic plate problem on

$$\Omega_\alpha^p = \omega_\alpha \times (-d, d), \quad \omega_\alpha = (0 < x_i < \alpha_i, i = 1, 2).$$

Here we will consider spaces of α -periodic functions in (x_1, x_2) with periods α_i in x_1, x_2 . A function $\phi(x_1, x_2, x_3)$ is called α -periodic if $\phi(x_1 + m\alpha_1, x_2 + n\alpha_2, x_3) = \phi(x_1, x_2, x_3)$ for arbitrary integers m, n . We define

$$\mathcal{H}^p(\alpha) = \{u_i \in (H_\alpha^1(\Omega_\alpha^p)), i = 1, 2, 3\} \quad (2.11)$$

where $H_\alpha^1(\Omega_\alpha^p)$ is the space of α -periodic functions in (x_1, x_2) and analogously we also define $L_{2,\alpha}(\omega_\alpha)$. The minimizer of $\mathcal{G}_{p,\alpha}^A(u)$ is now defined by (2.6) with the integration taken over Ω_α^p and ω_α , respectively. The solution of the α -periodic plate problem is the minimizer of $\mathcal{G}_{p,\alpha}^A(\omega)$ over $\mathcal{H}^p(\alpha)$. This minimizer exists if the q satisfies (2.9) (where the integral is now taken over ω_α) for all $\mathcal{R} \in \mathcal{N}^p \cap \mathcal{H}(\alpha) = \mathcal{N}_\alpha^p$ and is unique modulo \mathcal{N}_α^p . Once more the constraint (2.8) will be assumed here.

Together with the plate problem we will also consider its 2-dimensional counterpart which we will refer to as the beam problem. Its formulation can be obtained formally from that of the plate problem by the assumptions

$$\begin{aligned} q(x_1, x_2) &= q(x_1) \\ \frac{\partial u_i}{\partial x_2} &= u_2 = 0, \quad i = 1, 3. \end{aligned}$$

In this case we have

$$\Omega^b = \{x = (x_1, x_3) \mid |x_1| < L, |x_3| \leq d\} = \omega \times (-d, d) \quad (2.12)$$

where $\omega = I = (-L, L)$. The displacement vector is now $u = (u_1, u_3)$, $\sigma = \{\sigma_{ij}\}$, $e = \{e_{ij}\}$, $i, j = 1, 3$ and Hooke's law is given by

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \end{bmatrix} = \begin{bmatrix} \lambda + 2\mu & \lambda & 0 \\ \lambda & \lambda + 2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix} = \begin{bmatrix} e_{11} \\ e_{33} \\ 2e_{13} \end{bmatrix} \quad (2.13)$$

or

$$\sigma = Ae. \quad (2.14)$$

As before, for $q \in L_2(I)$, u is the minimizer of the quadratic functional

$$\mathcal{G}_b^A(u) = \frac{1}{2} \mathcal{E}_b^A(u) - \frac{1}{2} \int_{\omega} q(x_1)(u_3(x_1, d) + u_3(x_1, -d)) dx_1 \quad (2.15)$$

where $\mathcal{E}_b^A(u) = \int_{\Omega^b} e^T A e dx = \|u\|_E^2$ over a suitable space $\mathcal{H}(\Omega^b) \subset (H^1(\Omega^b))^2$ satisfying the constraint (2.8) for u_1, u_3 which characterizes the boundary conditions on the lateral sides. We denote as before by

$$\mathcal{N}^b = \{u(x_1, x_2) \mid u = (a_1, a_3) + b(x_3, -x_1), a_i, b \in R, i = 1, 3\} \quad (2.16)$$

the set of rigid body motions. Then if $\mathcal{N}_{\mathcal{H}(\Omega^b)}^b = \mathcal{N}^b \cap \mathcal{H}(\Omega^b) = \emptyset$, the solution exists and is unique. Otherwise we have to assume that

$$\int_I (0, q) \cdot \mathcal{R}^T dx_1 = 0 \quad \forall \mathcal{R} \in \mathcal{N}_{\mathcal{H}(\Omega^b)}^b. \quad (2.17)$$

In this case the minimizer exists and is unique modulo $\mathcal{N}_{\mathcal{H}(\Omega^b)}^b$.

If $\omega = R^1$ and $\Omega_{\infty}^b = R^1 \times (-d, d)$ then we assume that $q(x_1) \in L_2(R)$ has compact support and satisfies (2.17) for any $\mathcal{R} \in \mathcal{N}^b$. The minimizer of \mathcal{G}_b^A over

$$\mathcal{H}(\Omega_{\infty}^b) = \{u_i \in (H_{\text{loc}}^1(\Omega_{\infty}^b)), i = 1, 3 \mid \int_{\Omega_{\infty}^b} e^T A e dx < \infty\} \quad (2.18)$$

exists and is unique modulo \mathcal{N}^b as for the plate problem. We note that $\mathcal{H}(\Omega_{\infty}^b)$ is not a subspace of $(H^1(\Omega^b))^2$ because $u_i, i = 1, 3$ do not have to be square integrable on Ω^b .

Quite analogously as above we define the α -periodic beam problem. Here we set

$$\Omega_{\alpha}^b = \omega_{\alpha} \times (-d, d), \quad \omega_{\alpha} = (0 < x_1 < \alpha)$$

and will consider the space of α periodic functions (in x_1) on Ω_{α}^b . We define as before

$$\mathcal{H}^b(\alpha) = \{u_i \in H_{\alpha}^1(\Omega_{\alpha}^b), i = 1, 3\} \quad (2.19)$$

where $H_{\alpha}^1(\Omega_{\alpha}^b)$ is the space of α -periodic functions in x_1 with the condition (2.8). Analogously we define $L_{2,\alpha}(\omega_{\alpha})$. The functional $\mathcal{G}_{b,\alpha}^A(u)$ is defined by (2.15) where the integrals

are now taken over Ω_α^b and ω_α , respectively. Then the solution of the α -periodic beam problem is the minimizer of $\mathcal{G}_{b,\alpha}^A(u)$ over $\mathcal{H}^b(\alpha)$. This minimizer exists if q satisfies (2.17) (where the integral is to be taken over ω_α) for all $\mathcal{R} \in \mathcal{N}^p \cap \mathcal{H}^b(\alpha) = \mathcal{N}_\alpha^b$ and it is unique modulo \mathcal{N}_α^b .

Here and in what follows, we are using the super- and subscripts p and b to underline that we have in mind the plate or beam problem, respectively. If no ambiguity could occur, however, we shall simply omit these super- and subscripts.

2.2. Hierarchical Modelling of Plates and Beams

The fact that a parameter d which is small with respect to the diameter of ω suggests to replace the three-dimensional plate problem by a simpler, two-dimensional plate model. Since every plate model is only an approximation of the three-dimensional plate problem our goal is to construct a hierarchy of plate models (of increasing complexity) that are capable of approximating the solution of the three-dimensional plate problem for any fixed thickness d arbitrarily closely.

To this end we first introduce the notion of the n -model.

Definition 2.1 Let $n = (n_i), i = 1, 2, 3$ be a vector of nonnegative integers and $B(n) = \{b_{ij}\}$ be a (Hooke's) matrix of the type (2.2) which is assumed to be symmetric and positive definite and in general $B(n) \neq A$. Then we denote by ${}^B u^n(x)$ the solution of the n -plate model which is the unique minimizer of $\mathcal{G}_p^B(u)$ over the subspace $\mathcal{H}(n) \subset \mathcal{H}(\Omega^p)$ of displacement fields of the form

$${}^B u_i^n(x) = \sum_{j=0}^{n_i} U_{ij}(x_1, x_2) \left(\frac{x_3}{d} \right)^j. \quad (2.20)$$

For the beam the definition is analogous.

Existence and uniqueness of the solution of the n -model are proved along the same lines as for the exact solution. We can once more consider the bounded domain ω or the entire space R^2 for the plate (respectively R^1 for the beam).

Remark 2.1 The (material) matrix B used in the definition of the n -model may, and, in fact must be at times different from the matrix A of the plate problem, as we shall show below.

Remark 2.2 The approximation (2.20) is a classical one. The case $n = (1, 1, 0)$ was used for example in [10], [14], [40], $n = (1, 1, 2)$ in [17], [49], $n = (2, 2, 2)$ in [35] and $n = (3, 3, 2)$ in [31, 41]. In addition these forms have been used when a relation between U_{ij} (by differentiation) is imposed a priori, as e.g. in [46]. Many different approaches to derive the differential equations describing the model were suggested. We refer here for

example to [1, 2, 23, 24, 25, 30, 34, 37, 44]. The form (2.20) was also used in conjunction with various variational principles (direct or complementary) and additional heuristic ideas to obtain the differential equations describing the model. The exact solution to the three-dimensional plate problem and the error estimation have not been considered in these papers.

For the asymptotic error analysis called “justification” of various models we refer to [11, 12, 14, 15] and for an assessment and classification of various approaches as well as for additional references, see for example [12], and [7, 8, 33, 34]. The main question here is the analysis of the difference between the solution of the model and three dimensional solution as $d \rightarrow 0$. In these papers mainly the two sided estimates based on direct and complementary variational principles are used.

In the papers [9, 10, 16, 23, 24, 25, 30, 35, 36, 39, 49, 50] the form (2.20) is also suggested for laminated plates. In this context the following question arises:

Is the form (2.20) a heuristic one or can it be rigorously justified?

In the case of homogenous, isotropic material it was shown in [45] that (2.20) is not optimal for $d > 0$ and that a smaller error with respect to the three dimensional solution can be obtained with a model of the same complexity. Nevertheless, (2.20) is asymptotically optimal as $d \rightarrow 0$. For laminated plates, however, (2.20) is neither for $d > 0$ nor in any asymptotic sense optimal. An optimal form exists when the shape functions in x_3 belong to the kernel of a differential operator intrinsic to the problem, as has been shown in [47], [48]. For a concrete example of the hierarchical modelling of laminated plates along these lines we refer to [6].

Our approach uses the minimization of the total energy, i.e. the approximate solution is a projection. The merit of this method lies in that it can be very naturally and efficiently implemented within the frame work of the $h-p$ version of the finite element method which is available in the code MSC/PROBE. Hence the hierarchical modelling has to be understood as a numerical method for the solution of the original 3 dimensional problem with its singularities and boundary layers. The a-priori and a-posteriori error estimation as well as the adaptive model selection are essential here.

Since we consider only the problem of pure bending, the exact solution u^p of the plate problem satisfies (2.18). Correspondingly, we may set in (2.20)

$$\begin{aligned} U_{\alpha,j} = (x_1, x_2) = 0 & \text{ for } j \text{ even, } \alpha = 1, 2, \\ U_{3,j}(x_1, x_2) = 0 & \text{ for } j \text{ odd.} \end{aligned} \tag{2.21}$$

Since $u^p(x)$ is only an approximation to the solution $u^p(x)$ of the plate problem, we will judge the accuracy of the n -model by the relative error in a quantity of interest when a load $q \in \mathcal{P}$ is given. Here \mathcal{P} is a suitable class of loads which will be specified later.

For example, if we are interested in the error in energy, we define

$$E(d, \mathbf{n}, q) = \left(\frac{|\mathcal{E}^B(\mathbf{u}^{p,\mathbf{n}}) - \mathcal{E}^A(\mathbf{u}^p)|}{\mathcal{E}^A(\mathbf{u}^p)} \right)^{1/2} \quad (2.22)$$

where $\mathcal{E}^A(\mathbf{u}^p)$ resp $\mathcal{E}^B(\mathbf{u}^{p,\mathbf{n}})$ is the strain energy of \mathbf{u}^p respectively of $\mathbf{u}^{p,\mathbf{n}}$. Here \mathcal{P} could be the class of loads with compact support which satisfy (2.9). Then (2.22) is obviously well defined for any $q \in \mathcal{P}$.

Alternatively to (2.22), we can also consider the measure

$$E(d, \mathbf{n}, q) = \left(\frac{|\mathcal{E}^A(\mathbf{u}^{p,\mathbf{n}}) - \mathcal{E}^A(\mathbf{u}^p)|}{\mathcal{E}^A(\mathbf{u}^p)} \right)^{1/2} \quad (2.23)$$

where $\mathbf{u}^{p,\mathbf{n}}$ could be a “postprocessed” solution constructed from $\mathbf{u}^{p,\mathbf{n}}$.

For example we may be interested in

$$E(d, \mathbf{n}, q) = \frac{\left(\int_{R^2} (u_3^p(x_1, x_2, 0) - u^{p,\mathbf{n}}(x_1, x_2, 0))^2 dx_1 dx_2 \right)^{1/2}}{\left(\int_{R^2} (u_3^p(x_1, x_2, 0))^2 dx_1 dx_2 \right)^{1/2}} \quad (2.24)$$

where the class of loads \mathcal{P} is such that for a proper selection of the rigid body motion the numerator and the denominator are well defined for every $q \in \mathcal{P}$. We remark that the error measure in (2.22) - (2.24) are relative errors. They are natural in that in their analysis no assumptions on a scaling of the data q has to be made a priori. We will show below, however, that the accuracy of the model can be measured by analyzing only the numerator in (2.22) - (2.24).

Let us now define precisely what we mean by hierarchical models and give the framework in which we analyze the above error measures.

Definition 2.2 *A sequence of \mathbf{n} -models is hierarchical with respect to the measure E and the class \mathcal{P} of loads q if*

(H1) *(asymptotic exactness) for all $q \in \mathcal{P}$, $E(d, \mathbf{n}, q) \rightarrow 0$ as $d \rightarrow 0$ for all \mathbf{n} .*

(H2) *if \mathcal{P} is such that the exact solution \mathbf{u}^p is sufficiently smooth uniformly with respect to d then*

$$E(d, \mathbf{n}, q) \leq C(\mathbf{n})d^{\beta(\mathbf{n})}, \quad \beta(\mathbf{m}) \geq \beta(\mathbf{n}) \quad \text{for } m_i \geq n_i, i = 1, 2, 3.$$

The exponent $\beta(\mathbf{n})$ will be called rate of convergence.

(H3) *For any $q \in \mathcal{P}$ and any fixed $d > 0$ there holds $E(d, \mathbf{n}, q) \rightarrow 0$ as $\min\{n_i\} \rightarrow \infty$, $1 \leq i \leq 3$.*

(H4) *The plate model is monotonically hierarchical if for sufficiently large \mathbf{n} and $n_i \geq n_i$, $i = 1, 2, 3$, $E(d, \mathbf{m}, q) \leq E(d, \mathbf{n}, q)$.*

For the beam problem the definition is analogous.

Remark 2.3 It is clear that in our approach not only the solution of the plate models, but also the modelling itself (i.e. the selection of a particular model from the hierarchy) could be done numerically. This is in contrast to the approaches proposed in the references in Remark 2.2.

The following Theorem establishes the important properties (H3) and (H4) for our hierarchical models.

Theorem 2.1 *Let the \mathbf{n} -model be as in Definition 2.1 with $B(\mathbf{n}) = A$ for all \mathbf{n} . Then for the measures (2.22) or (2.23) and for ${}^A\bar{u}^{p,\mathbf{n}} = {}^A\bar{u}^{p,\mathbf{n}}$ the conditions (H3) and (H4) hold.*

Proof: ${}^A\bar{u}^{p,\mathbf{n}}$ is merely the energy projection of u^p onto $\mathcal{H}(\mathbf{n})$. This implies (H4) for (2.21). The density of the polynomials in $H^k(-d, d)$, $k = 1, 2$ yields (H3) for (2.22). Realizing that for ${}^A\bar{u}^{p,\mathbf{n}} = {}^A\bar{u}^{p,\mathbf{n}}$, we have $|\mathcal{E}^A({}^A\bar{u}^{p,\mathbf{n}}) - \mathcal{E}^A(u^p)| = \mathcal{E}^A({}^A\bar{u}^{p,\mathbf{n}} - u^p)$, and (H3) and (H4) for the measure (2.23) follow. \square

Remark 2.4 It is essential here that we formulate our models as energy projections and not in the form of asymptotic expansions which cease to make any sense if the three dimensional solution is not smooth. This, however, happens frequently because of the corners and edges that are typically present and because of the boundary layers in the three dimensional solution. In this situation many models as e.g. some of those in [2] are not usable.

We emphasize here that the hierarchical approach allows to employ simultaneously models of different order \mathbf{n} from the hierarchy in different parts of the plate. In particular in a neighborhood of the boundary of ω higher order models (in our sense) are essential, because among other things boundary layers are different for different models. see e.g. [4], [5].

Theorem 2.1 insures the desirable properties (H3) and (H4) for the \mathbf{n} -models and the measures (2.22), (2.23). Obviously these properties hold only for large \mathbf{n} and the proof sheds no light on the properties of the \mathbf{n} -models for small \mathbf{n} and d . In practice, however, we are especially interested in these lowest members of the hierarchy. These are in view of (2.20), (2.21) the $(1, 1, 0)$ -, $(1, 1, 2)$ - and the $(3, 3, 2)$ -model, respectively. In the following sections we therefore analyze these models in detail and show that they can be optimized in some sense.

3. Fourier Transform Solution of the Plate and Beam Problem and Their n-Models

3.1. The Fourier Transform of the Plate and Beam Problem

In Section 2 we defined the plate problem for $\omega = R^2$ and the beam problem for $\omega = R^1$ as well as the corresponding α -periodic problems.

In this section we will further elaborate on these two problems using the Fourier transformation as the main tool. Because we have to deal with functions which are not integrable we will employ the theory of Fourier transform of generalized functions (tempered distributions), see for example [18].

For any integrable function $q(x)$, $x \in R^n$ we denote by

$$\hat{q}(\xi) = \int_{R^n} q(x) e^{ix \cdot \xi} dx \quad \xi \in R^n \quad (3.1)$$

the Fourier transform of $q(x)$. Then $q(x)$ is defined by the inverse formula

$$q(x) = (2\pi)^{-n} \int_{R^n} \hat{q}(\xi) e^{-ix \cdot \xi} d\xi. \quad (3.2)$$

If $q(x) \in L_2(R^n)$ then $\hat{q}(\xi) \in L_2(R^n)$ and Parseval's formula

$$\int_{R^n} \phi \bar{\psi} dx = (2\pi)^{-n} \int_{R^n} \hat{\phi} \bar{\hat{\psi}} d\xi \quad (3.3)$$

holds.

If $q(x)$ is an $\alpha = \left(\frac{2\pi}{\beta_1}, \frac{2\pi}{\beta_2}, \dots, \frac{2\pi}{\beta_n} \right)$ periodic function and

$$q(x) = (2\pi)^{-n} \sum_m c_m e^{ix \cdot [m\beta]} \quad (3.4)$$

where $[m\beta] = (m_1\beta_1, m_2\beta_2, \dots, m_n\beta_n)$, $m = (m_1, m_2, \dots, m_n)$, m_i an integer, then in the sense of generalized functions

$$\hat{q}(\xi) = \sum_m c_m \delta(\xi + [m\beta]) \quad (3.5)$$

where $\delta(\xi)$ is the Dirac distribution.

The inverse transformation is then given by formula (3.4) and for any $r(x) = (2\pi)^{-n} \sum_m d_m e^{ix \cdot [m\beta]}$ Parseval's formula reads

$$\int_{0 < x_i < 2\pi/\beta_i} g \bar{r} dx = (2\pi)^{-n} (\beta_1 \beta_2 \dots \beta_n)^{-1} \sum_m c_m \bar{d}_m. \quad (3.6)$$

If $q(x) \in L_2(R^2)$ and has compact support then $\hat{q}(\xi)$ has all derivatives in a neighbourhood of $\xi = 0$ and the condition (2.9) reads

$$\hat{q}(0) = \frac{\partial \hat{q}}{\partial \xi_i}(0) = 0 \quad i = 1, 2. \quad (3.7)$$

For the periodic problem condition (2.9) yields

$$\hat{q}(0) = c_0 = 0. \quad (3.8)$$

Analogous results hold for the beam-problem.

In what follows we denote by $\hat{u}_i^p(\xi_1, \xi_2, x_3)$, $i = 1, 2, 3$, the partial Fourier transforms of the displacement components with respect to (x_1, x_2) and in the case of a beam we will write analogously $\hat{u}_i^b(\xi_1, x_3)$, with $i = 1, 3$.

Now using Parseval's formula we have

Theorem 3.1 *Let $u^p(x)$ resp $u^b(x)$ have finite strain energy. Then for the plate*

$$\begin{aligned} \mathcal{E}_p^A(u^p) &= \int_{\Omega_\infty^p} e^\top A e dx \\ &= \frac{E}{2(1+\nu)} \int_{R^2 \times (-d,d)} \left\{ \sum_{i=1}^3 |\nabla u_i|^2 + \frac{1}{1-2\nu} |\nabla \cdot u|^2 \right\} dx \\ &= (2\pi)^{-2} \frac{E}{2(1+\nu)} \int_{R^2 \times (-d,d)} \left\{ \sum_{i=1}^3 \left(|\xi|^2 |\hat{u}_i|^2 + |\hat{u}'_i|^2 \right) \right. \\ &\quad \left. \frac{1}{1-2\nu} |i\xi_1 \hat{u}_1 + i\xi_2 \hat{u}_2 + \hat{u}'_3|^2 \right\} d\xi_1 d\xi_2 dx_3. \end{aligned} \quad (3.9)$$

Here $|\xi|^2 = \xi_1^2 + \xi_2^2$ and the prime indicates $\frac{d}{dx_3}$. Further, for the beam

$$\begin{aligned} \mathcal{E}_b^A(u^b) &= (2\pi)^{-1} \frac{E}{2(1+\nu)} \int_{R^1 \times (-d,d)} \left\{ \xi_1^2 (|\hat{u}_1|^2 + |\hat{u}_3|^2) \right. \\ &\quad \left. + |\hat{u}'_1|^2 + |\hat{u}'_3|^2 + \frac{1}{1-2\nu} |i\xi_1 \hat{u}_1 + \hat{u}'_3|^2 \right\} d\xi_1 dx_3. \end{aligned} \quad (3.10)$$

For the periodic case when writing

$$\hat{u}_i = \left\{ c_m^{(i)}(x_3) \right\} \quad (3.11)$$

we get for the plate

$$\begin{aligned} \mathcal{E}_p^A(u_i^p) &= \int_{\{|x_i| < \alpha_i\} \times (-d,d)} e^\top A e dx \\ &= \frac{E}{2(1+\nu)} (2\pi)^{-2} \int_{-d}^d \sum_m \left\{ \sum_{\ell=1}^3 \left(|m|^2 |c_m^{(\ell)}|^2 + |c_m^{(\ell)\prime}|^2 \right) \right. \\ &\quad \left. + \frac{1}{1-2\nu} |im_1 c_m^{(1)\prime} + im_2 c_m^{(2)\prime} + c_m^{(3)\prime}|^2 \right\} dx_3 \end{aligned} \quad (3.12)$$

where $m = (m_1, m_2)$. For the beam we get analogously

$$\begin{aligned}\mathcal{E}_b^A(u^b) &= (2\pi)^{-1} \frac{E}{2(1+\nu)} \int_{-d}^d \sum_m \left\{ m^2 (c_m^{(1)}|^2 + |c_m^{(3)}|^2) \right. \\ &\quad \left. |c_m^{(1)\prime}|^2 + |c_m^{(3)\prime}|^2 + \frac{1}{1-2\nu} |imc_m^{(1)} + c_m^{(3)\prime}|^2 \right\} dx_3\end{aligned}$$

where m is an integer.

Proof: The proof is straightforward. We note that all terms are square integrable because we assumed that the strain energy is finite. \square

Let us also note that if the strain energy is finite then u is not necessarily square integrable on $R^2 \times (-d, d)$ not only for the possible presence of rigid body motions. Nevertheless all terms in the expressions for the strain energy are integrable.

Theorem 3.1 allows us to find the solution of the 3-dimensional formulation as well as that of any n-model explicitly in Fourier-transformed form.

Using the Fourier transformation of generalized functions, we now write down solution formulas for the beam and plate problems on the unbounded domain Ω_∞ . We start with the beam problem where $\xi \in R^1$ because, as will be seen in Theorem 3.3, the beam solution yields immediately the plate solution. It is not hard to prove:

Theorem 3.2

1. Let $q(x_1) \in L_2(R^1)$ have compact support and assume that

$$\int_{-\infty}^{\infty} q(x) dx = \int_{-\infty}^{\infty} x q(x) dx = 0.$$

Then the Fourier transform of the beam solution $u^b = (u_1^b, u_3^b)$ introduced in Section 2 is

$$\hat{u}^b = \begin{bmatrix} \hat{u}_1^b(\xi, x_3) \\ \hat{u}_3^b(\xi, x_3) \end{bmatrix} = \begin{bmatrix} \xi \psi(\xi, x_3) \\ \chi(\xi, x_3) \end{bmatrix} \hat{q}(\xi), \quad \xi \in R^1 \quad (3.13)$$

where

$$\begin{aligned}\psi(\xi, x_3) &= \frac{i}{2G\xi} \left\{ \sinh(\xi x_3) \left[\left(\nu - \frac{1}{2} \right) \frac{a}{\xi} + \frac{d}{2} b \right] - \frac{x_3}{2} \cosh(\xi x_3) a \right\} \\ \chi(\xi, x_3) &= \frac{1}{2G} \left\{ \cosh(\xi x_3) \left[(1 - \nu) \frac{a}{\xi} + \frac{d}{2} b \right] - \frac{x_3}{2} \sinh(\xi x_3) a \right\}\end{aligned} \quad (3.14)$$

with

$$a(\xi, d) = \frac{\cosh(\xi d)}{\cosh(\xi d) \sinh(\xi d) - \xi d}$$

and

$$b(\xi, d) = \frac{\sinh(\xi d)}{\cosh(\xi d) \sinh(\xi d) - \xi d}.$$

The solution is unique modulo the space of rigid body motions \mathcal{N}^b which, under Fourier transformation, becomes with the constraint (2.8)

$$\begin{aligned}\hat{\mathcal{N}}^b = \left\{ \hat{u}^b(\xi_1, x_3) = (\hat{u}_1^b(\xi_1, x_3), \hat{u}_3^b(\xi_1, x_3)) \mid \hat{u}_1^b(\xi_1, x_3) = ax_3\delta(\xi_1), \right. \\ \left. \hat{u}_3(\xi_1, x_3) = -ai\frac{\partial\delta}{\partial\xi_1} + b\delta(\xi_1), a, b \in R^1 \right\}\end{aligned}$$

where $\delta(\xi_1)$ is the Dirac distribution (in ξ_1).

2. Let $q(x) = e^{ix_1\xi_0}$, $\xi_0 \in R^1$, $\xi_0 \neq 0$. Then the $2\pi\xi_0^{-1}$ -periodic solution of the beam problem is

$$u^b = \begin{bmatrix} u_1^b(x_1, x_3) \\ u_3^b(x_1, x_3) \end{bmatrix} = \begin{bmatrix} \xi_0\psi(\xi_0, x_3) \\ \chi(\xi_0, x_3) \end{bmatrix} e^{ix_1\xi_0}, \quad (3.15)$$

and it is unique up to the rigid body motions

$$\mathcal{N}_{\text{PER}} = \{(u_1, u_3) \mid u_1 = 0, u_3 = a, a \in R^1\}.$$

Remark 3.1 In the special case of periodic solutions considered in Theorem 3.1 we did not write the solution u^b in its Fourier transformed form. Using the Fourier transformation, (3.15) can also be written in the form (3.13) with

$$\hat{q}(\xi) = 2\pi\delta(\xi + \xi_0).$$

Remark 3.2 The distributions \hat{u}_1 , \hat{u}_3 are in general not integrable since they have a singularity at the origin. Nevertheless it is easy to see that their strain energy in Theorem 3.1 is finite and hence the solution expressed by (3.13) is the Fourier transform of the unique (up to rigid body motions) solution introduced in Section 2. For more details see [45].

Remark 3.3 In Theorem 3.2 we have assumed a very special periodic load q . It is clear that we could represent a general periodic load by superposition.

So far we have dealt only with the beam problem. Now we will show that the Fourier solution formulas for the beam and plate problems are essentially the same. More precisely, we have

Theorem 3.3

1. Let $q(x_1, x_2) \in L_2(R^2)$ have compact support and satisfy (2.9). Then the Fourier transformation of the solution u^p of the plate problem is

$$\hat{u}^p(\xi_1, \xi_2, x_3) = \sqrt{2\pi}\hat{q}(\xi_1, \xi_2) \begin{bmatrix} \xi_1\psi(|\xi|, x_3) \\ \xi_2\psi(|\xi|, x_3) \\ \chi(|\xi|, x_3) \end{bmatrix} \quad (3.16)$$

where ψ, χ are as in Theorem 3.2 and the space of rigid body motions is now given by

$$\begin{aligned}\hat{\mathcal{N}}^p = \{u = (u_1(\xi_1, \xi_2, x_3), u_2(\xi_1, \xi_2, x_3), u_3(\xi_1, \xi_2, x_3)) \mid \\ u_1 = c_1 x_3 \delta(\xi_1, \xi_2), u_2 = c_2 x_3 \delta(\xi_1, \xi_2), u_3 = b \delta(\xi_1, \xi_2) - c_1 i \frac{\partial \delta}{\partial \xi_2} - c_2 i \frac{\partial \delta}{\partial \xi_1}\}\end{aligned}$$

2. For $q(x) = e^{i(x_1 \xi_{1,0} + x_2 \xi_{2,0})}$, $\xi_{1,0} \neq 0$, $\xi_{2,0} \neq 0$ we have

$$u^b = \begin{bmatrix} u_1^b(\xi_1, \xi_2, x_3) \\ u_2^b(\xi_1, \xi_2, x_3) \\ u_3^b(\xi_1, \xi_2, x_3) \end{bmatrix} = \sqrt{2\pi} \begin{bmatrix} \xi_{1,0} \psi((\xi_{1,0}^2 + \xi_{2,0}^2)^{1/2}, x_3) \\ \xi_{2,0} \psi((\xi_{1,0}^2 + \xi_{2,0}^2)^{1/2}, x_3) \\ \chi((\xi_{1,0}^2 + \xi_{2,0}^2)^{1/2}, x_3) \end{bmatrix} \cdot e^{i(x_1 \xi_{1,0} + x_2 \xi_{2,0})} \quad (3.17)$$

and

$$\mathcal{N}^p = \{u(\xi_1, \xi_2, x_3) = (u_1, u_2, u_3) \mid u_1 = 0, u_2 = 0, u_3 = a, a \in R^1\}.$$

Proof: First we observe that the functions ψ and χ given in (3.14) are even in ξ and hence the expressions $\psi(|\xi|, x_3)$, $\chi(|\xi|, x_3)$ have a natural meaning. Let us now consider the expression for the strain energy of the plate given in (3.9). Upon the substitution

$$\begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \end{bmatrix}(\xi_1, \xi_2, x_3) = \frac{\sqrt{2\pi}}{|\xi|} \begin{bmatrix} \xi_1 & \xi_2 \\ \xi_2 & -\xi_1 \end{bmatrix} \begin{bmatrix} \hat{v}_1 \\ \hat{v}_2 \end{bmatrix}(\xi_1, \xi_2, x_3)$$

$$\hat{u}_3(\xi_1, \xi_2, x_3) = \sqrt{2\pi} \hat{v}_3(\xi_1, \xi_2, x_3)$$

we get

$$\begin{aligned}\mathcal{E}_p^A(u) = (2\pi)^{-1} \frac{E}{2(1+\nu)} \int_{R^2 \times (-d, d)} |\xi|^2 (|\hat{v}_1|^2 + |\hat{v}_3|^2) + |\hat{v}'_1|^2 + |\hat{v}'_3|^2 \\ + \frac{1}{1-2\nu} |i|\xi| \hat{v}_1 + \hat{v}'_3|^2 d\xi_1 d\xi_2 dx_3 \\ + (2\pi)^{-1} \frac{E}{2(1+\nu)} \int_{R^2 \times (-d, d)} (|\xi|^2 |\hat{v}_2|^2 + |\hat{v}'_2|^2) d\xi_1 d\xi_2 dx_3.\end{aligned}$$

Comparing this with (3.10) and since $\mathcal{G}_b^A(u)$ is given by (2.6), we get $v_2 \equiv 0$ and obtain from Theorem 3.2

$$\begin{bmatrix} \hat{v}_1 \\ \hat{v}_3 \end{bmatrix}(\xi_1, \xi_2, x_3) = \hat{q}(\xi_1, \xi_2) \begin{bmatrix} |\xi| \psi(|\xi|, x_3) \\ \chi(|\xi|, x_3) \end{bmatrix}.$$

Hence we have

$$\begin{bmatrix} \hat{u}_1 \\ \hat{u}_3 \end{bmatrix} = \frac{\sqrt{2\pi}}{|\xi|} \begin{bmatrix} \xi_1 & \xi_2 \\ \xi_2 & -\xi_1 \end{bmatrix} \begin{bmatrix} \hat{v}_1 \\ \hat{v}_3 \end{bmatrix}$$

and this implies the statement of the theorem.

In the case of the periodic problem, i.e. for a load of the form $g = e^{i(x_1\xi_1+x_2\xi_2)}$, we utilize the fact that \hat{q} is the Dirac distribution. \square

In what follows we will assume that \hat{u}^b and \hat{u}^p are given by (3.13), (3.14), (3.15), (3.16), (3.17), i.e. we are fixing the rigid body motion.

3.2. The Fourier Transformation of the Solution of n-models

As we pointed out in Section 2, our main interest will focus on the lowest models of the hierarchy. Because the same relation between the Fourier transformation of the beam and plate problem holds also for the n-models, we will only address in detail the expression for the beam model.

For the $(1,1,0)$ model we use $B = B^{(1,1,0)} = B(\lambda_1, \mu_1, \kappa)$ where

$$B(\lambda_1, \mu_1, \kappa) = \begin{bmatrix} \lambda_1 + 2\mu_1 & \lambda_1 & \lambda_1 & 0 & 0 & 0 \\ \lambda_1 & \lambda_1 + 2\mu_1 & \lambda_1 & 0 & 0 & 0 \\ \lambda_1 & \lambda_1 & \lambda_1 + 2\mu_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \kappa\mu_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \kappa\mu_1 \end{bmatrix} \quad (3.18)$$

with (see (2.2), (2.4))

$$\lambda_1 = \frac{E_1 \nu_1}{(1 + \nu_1)(1 - 2\nu_1)}, \quad \mu_1 = \frac{E_1}{2(1 + \nu_1)} \quad (3.19)$$

$$\nu_1 = \frac{\nu}{1 + \nu}, \quad E_1 = \frac{E(1 + 2\nu)}{(1 + \nu)^2} \quad (3.20)$$

and $\kappa > 0$ is arbitrary.

For the n-models $\mathbf{n} = (n_1, n_2, n_3)$ with $n \geq 1, n_2 \geq 1, n_3 \geq 2$ we use $B^{\mathbf{n}} = B(\lambda, \mu, \kappa)$ where λ, μ are given by (2.4) and $\kappa > 0$ is again a free parameter, the so-called *shear correction factor*. Now we have the following representations of the solution.

Theorem 3.4 *For every \mathbf{n} and $B^{\mathbf{n}}$ as defined above, there exist functions*

$${}^B\psi^{\mathbf{n}}(\xi, x_3), {}^B\chi^{\mathbf{n}}(\xi, x_3)$$

so that the solution ${}^B u^{\mathbf{n}}$ of the n-model admits the representation (3.16),(3.17). In particular, we have

1. for the $(1, 0)$ model

$$B\psi^{(1,0)}(\xi, x_3) = -\frac{3i(1-\nu)}{4Gd^3\xi^4}x_3 \quad (3.22)$$

and

$$B\chi^{(1,0)}(\xi, x_3) = \frac{1}{4G} \left\{ \frac{3(1-\nu)}{d^3\xi^4} + \frac{2}{\kappa d\xi^2} \right\}, \quad (3.23)$$

2. for the $(1, 2)$ model

$$B\psi^{(1,2)}(\xi, x_3) = -\frac{3i}{4Gd^3} \frac{30(1-\nu) + d^2|\xi|^2(\kappa - 10\nu - 2\kappa\nu)}{(30 + d^2\kappa(1-\nu)|\xi|^2)|\xi|^4}x_3, \quad (3.24)$$

$$\begin{aligned} B\chi^{(1,2)}(\xi, x_3) = & \frac{3}{4Gd^3} \frac{30\kappa(1-\nu) + d^2|\xi|^2(20 + k^2 - 5\kappa\nu - 2\kappa^2\nu) - d^4\kappa(1-\nu)|\xi|^4}{\kappa(30 + d^2\kappa(1-\nu)|\xi|^2)|\xi|^4} \\ & + x_3^2 \frac{15}{4Gd^3} \frac{3\nu + d^2(\nu-1)|\xi|^2}{\kappa(\nu-1)d^2|\xi|^4 - 30|\xi|^2}, \end{aligned} \quad (3.25)$$

3. for the $(3, 2)$ model, as $|\xi| \rightarrow 0$,

$$B\psi^{(3,2)}(\xi, x_3) = -\frac{3i}{4Gd^3} \left\{ \left(\frac{1-\nu}{\xi^4} - \frac{(2+3\kappa\nu)d^2}{10\kappa\xi^2} \right) x_3 - \frac{2-\kappa\nu}{6k\xi^2} x_3^3 + O(d) \right\} \quad (3.26)$$

$$\begin{aligned} B\chi^{(3,2)}(\xi, x_3) = & \frac{3}{4Gd^3} \left\{ \frac{1-\nu}{\xi^4} + \frac{(8-3\kappa\nu)d^2}{10\kappa\xi^2} + \left(\frac{1-\nu}{5}d^2 - \frac{\nu}{2\xi^2} \right) x_3^2 \right. \\ & \left. - \frac{d^4(1-\nu)(21\kappa^2(1-2\nu) + (2-\kappa\nu)^2)}{525\kappa^2(1-2\nu)} \right\} + O(d^3\xi^2), \end{aligned} \quad (3.27)$$

and, as $|\xi| \rightarrow \infty$,

$$|\chi^{(3,2)}(\xi, x_3)| \leq C \frac{1}{|\xi|^2 d} \quad (3.28)$$

Proof: The formulas (3.22)–(3.28) are derived exactly in the same way as the formulae (3.14), (3.15). \square

Let us remark at this point that the functions ψ and χ are meromorphic in ξ in a neighborhood of $\xi = 0$.

The main idea is now to analyze the error of the n -model in the Fourier transform variables. We note that the matrix B^n is defined differently for $n_3 = 0$ and $n_3 \geq 2$. In both cases we included the shear factor κ and will analyze its optimal value which leads to the minimal error when the solution of the n -model is compared with the three-dimensional solution. We could also determine all the coefficients in B^n by the same principle. This, however, will lead, for the materials considered here, to our matrices B with κ being the only free parameter.

Remark 3.4 The introduction of the shear factor is a very old idea. It was introduced for the dynamic plate problem as well as for the static one. In [2, 40, 46] the shear factor $\kappa = 5/6$ is recommended. In [21, 32, 51] a value for the shear factor is obtained so that the velocity of short surface waves is modelled accurately. In [13, 16] the shear factor is designed so that an accurate solution will be obtained for a cantilever beam with a given cross section. Various other shear factors were suggested for laminated plates, e.g. [10, 35, 49, 50]. In [36], an adaptive construction for the shear factor is proposed. All these suggestions are ad hoc. For the $(1, 1, 0)$ model [28, 29, 38] analyze the optimal selection of the shear factor in connection with nonhomogenous lateral boundary conditions and $q = 0$ (the so-called Saint-Venant problem for plates) and show that for a particular value of κ the energy error is asymptotically of higher order.

In contrast to these derivations we will present in the next section an approach which leads to uniquely determined special values of the shear factor which in turn give a higher asymptotic rate of convergence for the data of intent in the plate bending problem.

4. Error Analysis and Optimal Shear Factors for the Hierarchical Models

The Fourier transformation allows us to analyze the performance of the n -models with respect to various measures of interest. In this section we will address some of them.

4.1. The Load Spaces and the Basic Performance Measures

In Section 2.2 we have introduced the accuracy measure relatively to a set of loads \mathcal{P} . Let us now define a specific set $\mathcal{P} = \mathcal{B}_{t,s}^l$ of loads $g(x_1, x_2)$ for the plate problem and $g(x_1)$ for the beam problem. Let for $t \geq 1$, $l \geq 0$ integers and $s \in \mathbb{R}^1$,

$$\begin{aligned} \mathcal{B}_{t,s}^l &= \{q \mid q(x) = 0, \text{ for } |x| \geq K, \int (1 + |\xi|^2)^s |\hat{q}(\xi)|^2 d\xi \leq 1, \\ &\quad (D^\alpha \hat{q})(0) = 0, \alpha = (\alpha_1, \alpha_2), 0 \leq \alpha_1, \alpha_2 \text{ integers } \alpha_1 + \alpha_2 = |\alpha|, |\alpha| \leq t, \\ &\quad \int_{|\xi| \leq 1} |\xi|^{-2+2(j-t)} |\hat{q}|^2 d\xi = A_j > 0, j = 0, 1, \dots, l\}. \end{aligned}$$

We note that from the basic properties of the Fourier transformation of distributions with compact support we have for $|\xi| \leq 1$, $|\alpha| \leq s$ that

$$|(D^\alpha \hat{q})(\xi)| \leq C(\alpha, s) \left(\int |\hat{q}|^2 (1 + |\xi|^2)^s d\xi \right)^{1/2}. \quad (4.1)$$

and hence, because $t \geq 1$, we obtain

$$\int_{|\xi| \leq 1} |\xi|^{-2(t+1)} |\hat{q}|^2 d\xi < \infty.$$

The space $\mathcal{B}_{t,s}^l$ is a space of generalized functions (distributions), based on which we introduce now the basic performance measures E corresponding to the data of interest Q . Obviously we could introduce many others. We have

a) *The energy error performance measure.*

$$E_E(B, \mathbf{n}, q) = \frac{\left| \int_{R^2} (\chi(|\xi|, d)) - {}^B\chi^{\mathbf{n}}(|\xi|, d)) |\hat{q}|^2 d\xi \right|^{1/2}}{\left| \int_{R^2} \chi(|\xi|, d) |\hat{q}|^2 d\xi \right|^{1/2}} \quad (4.2)$$

b) *The midsurface deflection measure.*

$$E_{\text{MID}}(B, \mathbf{n}, q) = \frac{\left| \int_{R^2} |\chi(|\xi|, 0) - {}^B\chi^{\mathbf{n}}(|\xi|, 0)|^2 |\hat{q}|^2 d\xi \right|^{1/2}}{\left| \int_{R^2} |\chi(|\xi|, 0)|^2 |\hat{q}|^2 d\xi \right|^{1/2}}. \quad (4.3)$$

c) *The average deflection measure.*

Let

$$\hat{u}_{3,\text{AV}}(\xi) = \frac{1}{2d} \int_{-d}^d \hat{u}_3(\xi, x_3) dx_3$$

and

$${}^B\hat{u}_{3,\text{AV}}^{\mathbf{n}}(\xi) = \frac{1}{2d} \int_{-d}^d \hat{u}_3^{\mathbf{n}}(\xi, x_3) dx_3.$$

Then

$$E_{\text{AV}}(B, \mathbf{n}, q) = \frac{\left| \int_{R^2} |\chi_{\text{AV}}(|\xi|) - {}^B\chi_{\text{AV}}^{\mathbf{n}}(|\xi|)|^2 |\hat{q}|^2 d\xi \right|^{1/2}}{\left| \int_{R^2} |\chi_{\text{AV}}(|\xi|)|^2 |\hat{q}|^2 d\xi \right|^{1/2}}. \quad (4.4)$$

The following theorem shows that in order to analyze the energy performance measure asymptotically as $d \rightarrow 0$ we need only consider the numerator of (4.2).

Theorem 4.1 *Let $q \in \mathcal{B}_{1,s}^l$, $s \geq -1/2$, $l \geq 0$, $0 < d \leq d_0 \leq 1$. Then*

$$\int_{R^2} \chi(|\xi|, d) |\hat{q}(\xi)|^2 d\xi \geq C d^{-3} \quad (4.5)$$

where $C > 0$ is independent of d .

Proof: First we observe that it follows from (3.14) that for $|\xi|d \geq 1$

$$|\chi(|\xi|, d)| \leq C_1 \frac{1}{|\xi|}, \quad (4.6)$$

with C_1 independent of d . Further we have for all $|\xi|d \leq 1$

$$\chi(|\xi|, d) = a_0 d^{-3} |\xi|^{-4} + R(\xi, d) \quad \text{where} \quad |R(\xi, d)| \leq C_2 d^{-1} |\xi|^{-2} \quad (4.7)$$

and C_2 is independent of d and $a_0 \neq 0$. Now we write

$$\int_{R^2} \chi(|\xi|, d) |\hat{q}|^2 d\xi = \int_{|\xi|d \geq 1} + \int_{|\xi| \leq 1} + \int_{|\xi| > 1, |\xi|d < 1}$$

and consider the three terms separately. Using (4.6) we get for $s \geq -1/2$ that

$$\begin{aligned} \left| \int_{|\xi|d \geq 1} \chi(|\xi|, d) |\hat{q}|^2 d\xi \right| &\leq \left| \int_{|\xi|d \geq 1} \chi(|\xi|, d) (1 + |\xi|^2)^{-s} (1 + |\xi|^2)^s |\hat{q}|^2 d\xi \right| \\ &\leq \sup_{|\xi|, |\xi|d \geq 1} |\chi(|\xi|, d) (1 + |\xi|^2)^{-s}| \leq C. \end{aligned}$$

Further

$$\int_{|\xi| \leq 1} \chi(|\xi|, d) |\hat{q}|^2 d\xi = a_0 d^{-3} \int_{|\xi| \leq 1} |\xi|^{-4} |\hat{q}|^2 d\xi + \int_{|\xi| \leq 1} R(\xi, d) |\hat{q}|^2 d\xi.$$

As $q \in \mathcal{B}_{1,s}^0$, we have

$$a_0 d^{-3} \int_{|\xi| \leq 1} |\xi|^{-4} |\hat{q}|^2 d\xi = a_0 d^{-3} A_0$$

and

$$\left| \int_{|\xi| \leq 1} R(\xi, d) |\hat{q}|^2 d\xi \right| \leq d^{-1} C.$$

Hence for $d \leq d_0$ we get

$$\int_{|\xi| \leq 1} = a_0 d^{-3} (A_0 + O(d^2)).$$

Finally

$$\int_{|\xi| \geq 1, |\xi|d \leq 1} \chi(|\xi|, d) |\hat{q}|^2 d\xi = a_0 d^{-3} \int_{|\xi| \geq 1, |\xi|d \leq 1} |\xi|^{-4} |\hat{q}|^2 d\xi + \int_{|\xi| \geq 1, |\xi|d \leq 1} R(\xi, d) |\hat{q}|^2 d\xi$$

where

$$a_0 d^{-3} \int_{|\xi| \geq 1, |\xi|d \leq 1} |\xi|^{-4} |\hat{q}|^2 d\xi = a_0 d^{-3} \rho, \quad \rho \geq 0$$

and

$$\begin{aligned} \left| \int_{|\xi| \geq 1, |\xi|d \leq 1} R(\xi, d) |\hat{q}|^2 d\xi \right| &\leq C d^{-1} \int_{|\xi| \geq 1, |\xi|d \leq 1} \frac{1}{|\xi|^2} (1 + |\xi|^2)^{-s} (1 + |\xi|^2)^s |\hat{q}|^2 d\xi \\ &\leq \bar{C} d^{-1}. \end{aligned}$$

Hence collecting all the terms we get for $0 < d < d_0$

$$\left| \int_{R^2} \chi(|\xi|, d) |\hat{q}|^2 d\xi \right| \geq |a_0| d^{-3} (1 + O(d^2))$$

which yields (4.5). \square

In what follows we will also need the behaviour of χ, χ_{AV} for large $|\xi|$. We have

Theorem 4.2

1. For $|\xi|d \geq 1$,

$$|\chi_{AV}(|\xi|)| \leq \frac{C}{|\xi|^2 d} \quad (4.8)$$

$$|\chi(|\xi|, 0)| \leq \frac{C(m)}{|\xi|^m d^{m-1}}, \quad m \geq 0 \quad \text{arbitrary.} \quad (4.9)$$

2. For $|\xi|d \leq 1$,

$$\chi_{AV}(\xi, d) = a_0 d^{-3} |\xi|^{-4} + R(\xi, d) \quad (4.10)$$

where

$$|R| \leq C d^{-1} |\xi|^{-2}, a_0 \neq 0,$$

and

$$\chi(|\xi|, 0) = \tilde{a}_0 d^{-3} |\xi|^{-4} + R, \quad (4.11)$$

$$|R| \leq C d^{-1} |\xi|^{-2}, \tilde{a}_0 \neq 0.$$

Proof: (4.9) and (4.11) follow directly from (3.14).

By an easy calculation we obtain from (3.14) also that

$$\begin{aligned} \chi_{AV}(\xi) &= \frac{1}{2d} \int_{-d}^d \chi(\xi, x_3) dx_3 \\ &= \frac{1}{G} \frac{(3-2\nu)(1-e^{4\xi d}) + 4\xi d e^{2\xi d}}{4\xi^2 d(1-e^{4\xi d}) + 16\xi^3 d^2 e^{2\xi d}} \end{aligned} \quad (4.12)$$

from which (4.10) and (4.11) follow. \square

Now we can prove along the lines of Theorem 4.1 the following result for χ_{AV} and $\chi(\xi, 0)$.

Theorem 4.3 Let $g \in \mathcal{B}_{3,s}^l$, $l \geq 0$, $s \geq -2$ and $0 < d \leq d_0 < 1$. Then

$$\int |\chi_{AV}(\xi)|^2 |\hat{q}|^2 d\xi \geq C d^{-6}, \quad (4.13)$$

and for every real s we have

$$\int |\chi(\xi, 0)|^2 |\hat{q}|^2 d\xi \geq C(s) d^{-6}, \quad (4.14)$$

where $C > 0$ is independent of d .

Theorems 4.1 and 4.3 allow us to analyze only the numerators in (4.2), (4.3) and (4.4) in order to prove the conditions (H1) and (H2) in Definition 2.2. Since the analysis of all error measures proceeds along similar lines, we present it here only for (4.2) in detail and elaborate briefly on (4.3), (4.4). The following theorem shows that the asymptotic behaviour of the modelling error for the n -model is completely governed by the behaviour of ${}^B\chi^n(\xi, d)$ near $|\xi| = 0$, provided the data is sufficiently smooth.

Theorem 4.4 *Assume that for $|\xi|d \leq 1$ and every integer $k \geq 0$*

$$\chi(|\xi|, d) = \sum_{j=0}^k a_j |\xi|^{-4+2j} d^{-3+2j} + R_k \quad (4.15)$$

with

$$|R_k| \leq C d^{-3+2(k+1)} |\xi|^{-4+2(k+1)} \quad (4.16)$$

and analogously that

$${}^B\chi^n(|\xi|, d) = \sum_{j=0}^k {}^B a_j^n |\xi|^{-4+2j} d^{-3+2j} + {}^B R_k^n \quad (4.17)$$

with

$$|{}^B R_k^n| \leq C d^{-3+2(k+1)} |\xi|^{-4+2(k+1)}. \quad (4.18)$$

In addition assume that for $|\xi|d \geq 1$

$$|\chi(\xi, d) - {}^B\chi^n(\xi, d)| \leq C \frac{1}{|\xi|} \quad (4.19)$$

where C is independent of d . Let further

$$j^* = \min_j \{a_j \neq {}^B a_j^n\} \quad (4.20)$$

where the minimum is taken over all nonnegative integers.

Then for $q \in \mathcal{B}_{1,s}^l$, $s \geq -1/2$, $l \geq j^*$ we get that

$$\lim_{d \rightarrow 0} E_E(B, n, q) d^{-\beta} = C, \quad 0 < C < \infty \quad \text{for } \beta = \min\{s + 2, j^*\}. \quad (4.21)$$

Proof: Using Theorem 4.1 we have to prove that

$$\begin{aligned} Z &= \left| \int (\chi(|\xi|, d) - {}^B\chi^n(|\xi|, d)) |\hat{q}|^2 d\xi \right| \\ &= C d^{2\beta-3} (1 + o(1)) \end{aligned}$$

as $d \rightarrow 0$. To this end we write as in the proof of Theorem 4.1

$$Z = \int_{|\xi|d \geq 1} + \int_{|\xi| \leq 1, |\xi|d \geq 1} + \int_{|\xi| \geq 1, |\xi| < 1}.$$

Now using (4.19) we get for $s \geq -1/2$

$$\begin{aligned} \int_{|\xi|d \geq 1} (\chi(|\xi|, d) - {}^B\chi^{\mathbf{n}}(|\xi|, d)) |\hat{q}|^2 d\xi &\leq \sup_{|\xi|, |\xi|d \geq 1} C(1 + |\xi|^2)^{-s} \frac{1}{|\xi|} \\ &\leq Cd^{2s+1}. \end{aligned}$$

Further,

$$\int_{|\xi| \leq 1, |\xi|d \leq 1} = (a_{j^*} - {}^B a_{j^*}^{\mathbf{n}}) d^{-3+2j^*} A_{j^*} + R$$

where $|R| \leq Cd^{-3+2(j^*+1)}$ and finally

$$\int_{|\xi| \geq 1, |\xi|d \leq 1} = (a_{j^*} - {}^B a_{j^*}^{\mathbf{n}}) d^{-3+2j^*} + R$$

where $|R| \leq Cd^{-3+2(j^*+1)}$ and therefore (4.21) is proven. \square

Analogously we can prove

Theorem 4.5 Suppose that for $\chi_{AV}(|\xi|)$, ${}^B\chi_{AV}^{\mathbf{n}}(|\xi|)$, $\chi(|\xi|, 0)$ and ${}^B\chi^{\mathbf{n}}(|\xi|, 0)$ we have expansions analogous to (4.15)-(4.18) for $|\xi|d \leq 1$ and that for $|\xi|d \geq 1$ there holds

$$|\chi_{AV}(|\xi|) - {}^B\chi_{AV}^{\mathbf{n}}(|\xi|)| \leq \frac{C}{|\xi|^2 d}, \quad (4.22)$$

$$|\chi(|\xi|, 0) - {}^B\chi^{\mathbf{n}}(|\xi|, 0)| \leq \frac{C}{|\xi|^2 d}. \quad (4.23)$$

Then, if $q \in \mathcal{B}_{3,s}^l$ with $s \geq -2$ and $l \geq 2j^* - 2$, we have

$$\lim_{d \rightarrow 0} E_{AV}(B, \mathbf{n}, q) d^{-\beta} = C \quad 0 < C < \infty, \quad \text{with } \beta = \min\{s + 4, 2j^*\}. \quad (4.24)$$

Here j^* is defined as in (4.20). For the error measure E_{MID} defined in (4.3) the same bound as in (4.24) holds.

The proof Theorem 4.5 is analogous to that of Theorem 4.4 and uses Theorem 4.2.

From Theorems 4.4 and 4.5 we see in particular that the \mathbf{n} -model is asymptotically exact only if $j^* \geq 1$.

4.2. Asymptotic Analysis for n-Models

We shall now apply theorems 4.3 and 4.4 to the analysis of the n-models and, in particular, to the determination of the optimal shear correction factors κ .

4.2.1. The Reissner - Mindlin — (1, 1, 0) Model

We have from Theorem 3.2 that for $|\xi|d < 1$

$$\begin{aligned} \chi(|\xi|, x_3) = & \frac{1}{4G} \left(\frac{3(1-\nu)}{|\xi|^4 d^3} - \frac{3\nu x_3^2}{2|\xi|^2 d^3} + \frac{3(8-3\nu)}{10|\xi|^2 d} + \right. \\ & \left. \frac{(157\nu - 227)d^4 + (1050 - 630\nu)d^2 x_3^2 - 175(1+\nu)x_3^4}{1400d^3} \right) + O(d^3|\xi|^3) \end{aligned} \quad (4.25)$$

and

$$\chi_{AV}(|\xi|) = \frac{1}{4G} \left(\frac{3(1-\nu)}{d^3|\xi|^4} + \frac{12-7\nu}{5d|\xi|^2} + \frac{11d(1-\nu)}{175} \right) + O(d^3|\xi|^2). \quad (4.26)$$

On the other hand using Theorem 3.4, we get for $n = (1, 1, 0)$

$$\mathcal{B}\chi^{(1,1,0)}(|\xi|, x_3) = \frac{1}{4G} \left\{ \frac{3(1-\nu)}{d^3|\xi|^4} + \frac{2}{\kappa d|\xi|^2} \right\}. \quad (4.27)$$

By (4.7) and (4.27) we see that (4.19) is satisfied and furthermore, comparing (4.27) with (4.25), we find for the measure E_E that:

If $q \in \mathcal{B}_{1,s}^l$ for $l \geq 1$ and $s \geq -1/2$ then

$$\beta = \begin{cases} \min\{2, s+2\} & \text{if } \kappa = \kappa_{\text{opt}} = \frac{5}{6(1-\nu)} \\ 1 & \text{if } \kappa \neq \kappa_{\text{opt}} \end{cases}. \quad (4.28)$$

For the measure E_{MID} we obtain with Theorem 4.5 in the same fashion:

If $q \in \mathcal{B}_{3,s}^l$ for $l \geq 2$ and $s \geq -2$ then

$$\beta = \begin{cases} \min\{4, s+4\} & \text{if } \kappa = \kappa_{\text{opt}} = \frac{20}{3(8-3\nu)} \\ 2 & \text{if } \kappa \neq \kappa_{\text{opt}} \end{cases}. \quad (4.29)$$

For the measure E_{AV} we get:

If $g \in \mathcal{B}_{3,s}^l$, $l \geq 2$ and $s \geq -2$, then

$$\beta = \begin{cases} \min\{4, s+4\} & \text{if } \kappa = \kappa_{\text{opt}} = \frac{10}{12-7\nu} \\ 2 & \text{if } \kappa \neq \kappa_{\text{opt}} \end{cases}. \quad (4.30)$$

Let us comment on these results. We see that the optimal value of κ depends on the goal of the plate modelling. For $\nu = 0$ we get in all cases that $\kappa_{\text{opt}} = 5/6$ i.e. the classical value proposed for example in [2, 40, 46]. Secondly, for data q which is not sufficiently smooth (or equivalently, for the case when the solution of the three-dimensional problem lacks regularity), i.e. when s is small, nothing can be gained by the above choices of κ . We will analyze this case separately below. Finally we remark that for the choice $B = A$ the $(1, 1, 0)$ model is not asymptotically exact, i.e. the use of the modified material matrix $B^{(1,1,0)}$ in (3.18) is necessary for the asymptotic exactness of the $(1, 1, 0)$ model.

4.2.2. The $(1, 1, 2)$ Model

From (3.26), (3.27) and (4.26) we find estimates which are identical to (4.28)-(4.30) provided that the optimal shear correction factor is given by

$$\kappa_{\text{opt}} = \frac{12 - 2\nu}{\nu^2} \left\{ -1 + \sqrt{1 + \frac{20\nu^2}{(12 - 2\nu)^2}} \right\}, \quad \nu \neq 0. \quad (4.31)$$

We point out that unlike for the $(1, 1, 0)$ -model here κ_{opt} is the same for all three accuracy measures used. In Table 1 we report the optimal shear correction factor for various values of ν .

Table 1 The values of κ_{opt} for the $(1, 1, 2)$ model.

ν	κ_{opt}
0	5/6
0.25	0.867
0.30	0.874
0.40	0.887
0.50	0.899

For other values of the Poisson ratio ν it could be easily computed from (4.31).

Remark 4.1 For $\kappa = 1$ the measure E_E is also the relative error in the energy norm. For $\kappa \neq 1$ the error in the energy norm has to be computed separately and we shall not elaborate on it here.

4.2.3. The $(3, 3, 2)$ Model

Reasoning as for the $(1, 1, 2)$ model we obtain now that

For the error measure E_E : If $q \in \mathcal{B}_{1,s}^l$ with $s \geq -1/2$ and $l \geq 2$, then

$$\beta = \begin{cases} \min\{2, s+2\} & \text{if } \kappa = \kappa_{\text{opt}} = 1, \\ 1 & \text{if } \kappa \neq \kappa_{\text{opt}} \end{cases} . \quad (4.32)$$

For the error measures E_{AV} and E_{MID} we find: If $q \in \mathcal{B}_{3,s}^l$ with $s \geq -2$ and $l \geq 2$, then

$$\beta = \begin{cases} \min\{4, s+4\} & \text{if } \kappa = \kappa_{\text{opt}} = 1, \\ 2 & \text{if } \kappa \neq \kappa_{\text{opt}} \end{cases} .$$

Consequently, for sufficiently smooth loads q , the introduction of shear correction factors $\kappa \neq 1$ into the \mathbf{n} -models with $\mathbf{n} \geq (3, 3, 2)$ componentwise will always decrease the rate of convergence and hence the choice $B^{\mathbf{n}} = A$ is optimal in these cases. We further see that the rate of convergence β achieved by the $(3, 3, 2)$ model is the same as for the $(1, 1, 0)$ model with κ_{opt} . For sufficiently large s we get a higher rate of convergence only by using the $(3, 3, 4)$ model. Then we can replace 2 and 4 in the expressions for β by 3 and 6, respectively (see [45]).

4.3. The Error Analysis for Unsmooth Loads

In the previous sections we analyzed the behaviour of the plate modelling error as $d \rightarrow 0$ for a class of loads q . We have seen that the accuracy of the \mathbf{n} model was in this case governed by the behaviour of the functions ψ and χ for small ξ .

Furthermore, the smoothness of the load q in the class $\mathcal{B}_{t,s}^l$ was characterized by the parameter s which in turn characterized the decay of $|\hat{q}|$ as $|\xi| \rightarrow \infty$. In the proof of Theorem 4.3 and Theorem 4.5 the minimal allowable s was governed by the decay of $(\chi - {}^B\chi^{\mathbf{n}})(\xi)$ for large ξ . Therefore we could be interested in the selection of the shear correction factor, for which this decay will be maximal.

Theorem 4.6 *Consider the $(1, 1, 0)$ model. Then we have for $|\xi|d \geq 1$ that*

$$|\chi_{AV}(\xi) - {}^B\chi_{AV}^{\mathbf{n}}(\xi)| \leq \frac{C}{|\xi|^4 d^3} \text{ for } \kappa = \frac{2}{3-2\nu} = \kappa_{\text{opt}} \quad (4.33)$$

$$\frac{C_1}{d|\xi|^2} \leq |\chi_{AV}(\xi) - {}^B\chi_{AV}^{\mathbf{n}}(\xi)| \leq \frac{C_2}{|\xi|^2 d} \text{ for } \kappa \neq \kappa_{\text{opt}} \quad (4.34)$$

where C, C_1, C_2 are positive constants which are independent of d .

Proof: Using (4.12) and (4.27), it is easy to show that there exist $C_1, C_2 > 0$ so that

$$C_1 \frac{1}{|\xi|^4 d^3} \leq \left| \chi_{AV}(\xi) - {}^B\chi_{AV}^{\mathbf{n}}(\xi) - \frac{1}{2} \left(\frac{3-2\nu}{2} - \frac{1}{\kappa} \right) \frac{1}{\xi^2 d} \right| \leq C_2 \frac{1}{|\xi|^4 d^3}$$

from which (4.33) and (4.34) follow. \square

This theorem can now be used to obtain optimal shear correction factors for the error measure E_{AV} and certain classes of data q . To this end let us assume that the thickness d is fixed. Then, based on Theorem 4.6 we can construct a sequence of loads q_k , $k = 1, 2, \dots$, so that

$$\int_{R^2} |\chi_{AV}|^2 |\hat{q}_k|^2 d\xi < \infty,$$

$$\int_{R^2} |\chi_{AV}|^2 |\hat{q}_k|^2 d\xi \rightarrow \infty \quad \text{as } k \rightarrow \infty$$

with $q_k \in B_{3,s}^l$, where $l \geq 0$ and $-4 \leq s \leq -2$ and for which

$$E_{AV}(B, n, q_k) \rightarrow 0 \quad \text{for } \kappa = \kappa_{\text{opt}} = \frac{2}{3 - 2\nu},$$

and

$$E_{AV}(B, n, q_k) \neq 0 \quad \text{for } \kappa \neq \kappa_{\text{opt}}.$$

In Table 2 we show the values κ_{opt} from (4.30) for "smooth" q and (4.33) for "unsmooth" q for the $(1, 1, 0)$ - model and the error measure E_{AV} .

Table 2 Comparison of κ_{opt} of (4.30) and (4.33) for E_{AV} and the $(1, 1, 0)$ -model.

ν	(4.30)	(4.33)
0	0.83	0.67
0.25	0.98	0.80
0.30	1.01	0.83
0.40	1.09	0.91
0.50	1.13	1.00

Table 2 shows that in the case of average displacement we take for less smooth q the smaller shear factor. We emphasize that this conclusion is valid only for the average normal deflection measure E_{AV} .

Remark 4.2 We point out that for the measures E_{MID} and E_E the optimal shear correction factors κ_k for the above sequence of loads must become large and positive as $\kappa \rightarrow \infty$. This follows from a comparison of (4.27) and (3.14).

In section 3 we have introduced the periodic solution with the load $q = e^{ix\xi}$, $x = (x_1, x_2)$. In this case the solutions of the three dimensional problem and the n -model have the same form in (x_1, x_2) . Consequently, we can easily compute the shear factor κ as a function of $|\xi_0|d$ which yields the same result for the exact and n -model solution for the given quantity of interest, for example for the measure E_{AV} . The graphs of the

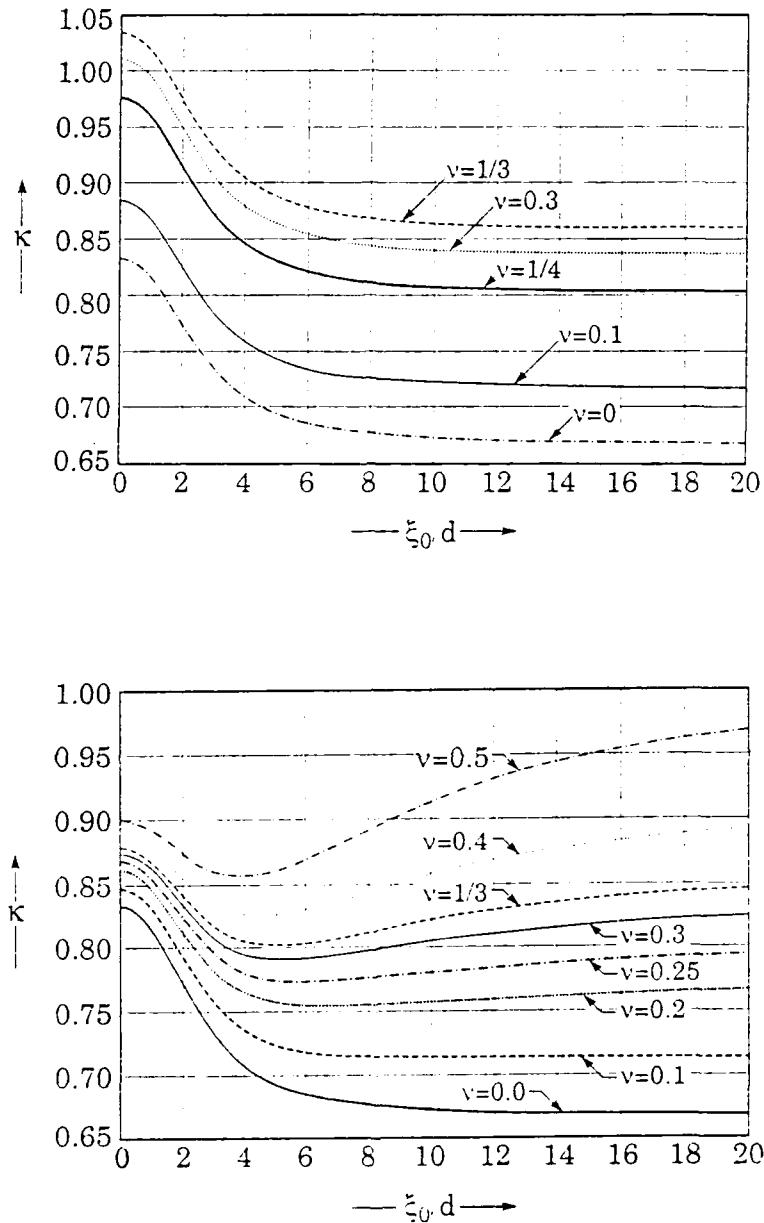


Figure 1: Optimal shear correction factors for the $(1,1,0)$ model (top) and the $(1,1,2)$ model (bottom) with periodic loads as functions of $|\xi_0|d$.

optimal shear correction factors for the $(1, 1, 0)$ and $(1, 1, 2)$ model are given in Figure 1. We note that in the limit $|\xi_0|d \rightarrow 0$ we obtain κ_{opt} from (4.30), while for $|\xi_0|d \rightarrow \infty$ we get κ_{opt} from (4.33).

In summary, we saw in this section that the shear correction factor for any n -model can be uniquely determined by some optimality requirement and we have analyzed some of the possible choices for this requirement. We found that above a certain threshold of the model order, the introduction of a shear correction factor into the plate model does not improve the asymptotic rate of convergence. For the homogeneous and isotropic materials considered here only the $(1, 1, 0)$ and the $(1, 1, 2)$ models can be improved by a judicious choice of κ . In addition, the optimal shear correction factors also depend on the regularity of the three-dimensional solution which is approximated by the plate model. We point out that many error measures different from the one considered here could be analyzed in the same fashion. It is interesting to note that for different goals of the plate modelling as well as for different regularity properties of the three dimensional solution different shear factors are recommendable. This also suggests that in the neighborhood of the boundary different shear factors should be used than in the interior since boundary layers or singularities of the three dimensional solution govern the quality of the plate model there. We will also see this in the numerical experiments in the following section.

5. Numerical Experiments

In the previous sections we obtained several optimal values for the shear correction factor κ for the infinite beam and plate problem and for different quantities Q of interest.

For the problems in applications the domain is bounded. In this case the theoretical convergence of the various models can be analyzed as in [7] where the model $(1, 1, 0)$ was addressed. Then also the solution is usually not smooth because, for example, of various boundary layers caused by the incompatibility of the data and the lateral boundary conditions, see e.g. [4, 5]. Hence we will now investigate the influence of κ for the following model problem.

Consider a *clamped* beam of thickness $2d = 0.2$ and length 2 with Youngs modulus $E = 1$ and Poisson ratio $\nu = 0.3$ as in Figure 2 where the load $q(x)$ is also defined. Because of the obvious symmetry in the problem we do the computation for the half beam only. First, as exact solution we will use the numerical solution by the $h-p$ finite element method obtained with the program MSC/PROBE with high polynomial degree $p = 8$ and refined meshes near $x_1 = 0.2$, $x_1 = 0.5$ and $x_3 = \pm d$. The relative error in the energy norm was less than 10^{-8} so that this solution could be considered as the exact one.

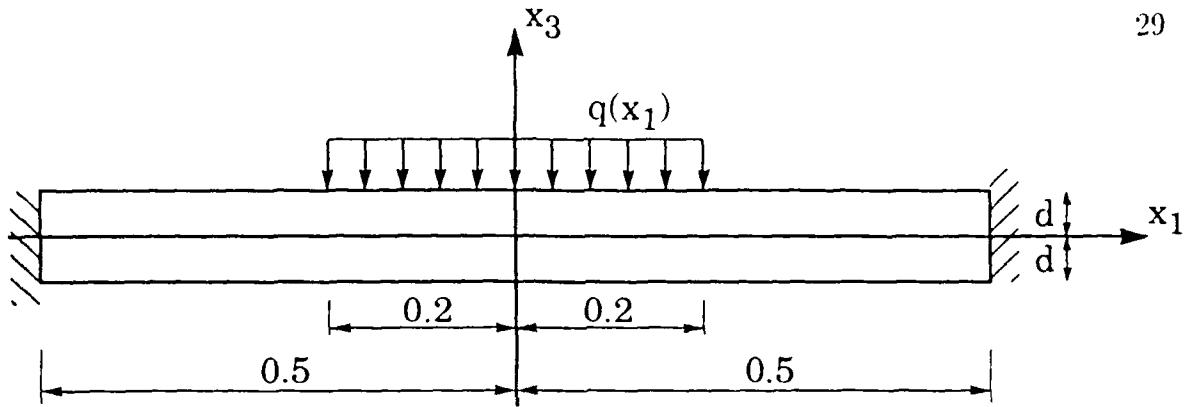


Figure 2: Scheme of the model problem

Then the n -model of the beam problem leads to a two point boundary value problem for a system of ordinary differential equations. This boundary value problem was in turn solved numerically to high accuracy by the method described in [3] and also its solution could be taken as the exact solution. Hence the difference between the two solutions is the modelling error as discussed in the previous sections.

Figure 3 shows the pointwise error in the average normal deflection for the $(1,0)$ -model. The curves correspond to (from above) $\kappa = 1.0, 0.938$ (the theoretically optimal value for the midsurface deflection from (4.29) 0.935, 0.87 (suggested in [13]) and $\kappa = 5/6$ which was suggested in [2, 40, 46].

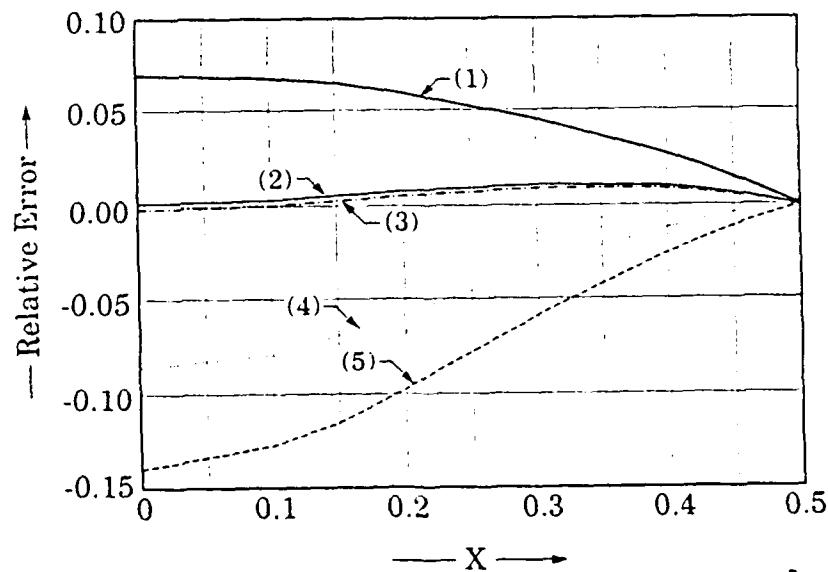


Figure 3: Pointwise error in the average normal deflection for the $(1,0)$ -model.

- (1) $\kappa = 1$
- (2) $\kappa = 0.938$ (optimal value (4.29) for the midfiber)

- (3) $\kappa = 0.935$ (optimal numerical value)
- (4) $\kappa = 0.87$ (Cowper [13])
- (5) $\kappa = 5/6$ (Reissner [40])

Clearly the results obtained with $\kappa = 1$ exhibit a large error which could be expected. Surprisingly, however, the values proposed in [13] and [2, 40, 46] are not leading to a smaller error, either. On the other hand, the value $\kappa = 0.935$ which was found numerically as an optimal value for the given case is very close to the value $\kappa = 0.938$ which is, according to (4.29), optimal for the *midsurface* deflection. We point out that $\kappa = 0.935$ is not close to the value $\kappa = 1.01$ in (4.30) respectively $\kappa = 0.83$ in (4.33). The optimal value is obviously in between. This effect is caused by the fact that the solution is not smooth uniformly through the beam. Consequently, (4.33) suggests the value $\kappa = 0.833$ near $x_1 = 0.5$ and by (4.30) $\kappa = 1.01$ elsewhere. Because of the boundary layer we select therefore

$$\kappa(x) = 1.01 + (0.833 - 1.01)e^{-70(x-0.5)^2}. \quad (5.1)$$

In Figure 4 we show the error of the $(1, 0)$ -model for the average (through the thickness) normal deflection for the numerically optimal value $\kappa = 0.93$ and for $\kappa(x)$ given by (5.1).

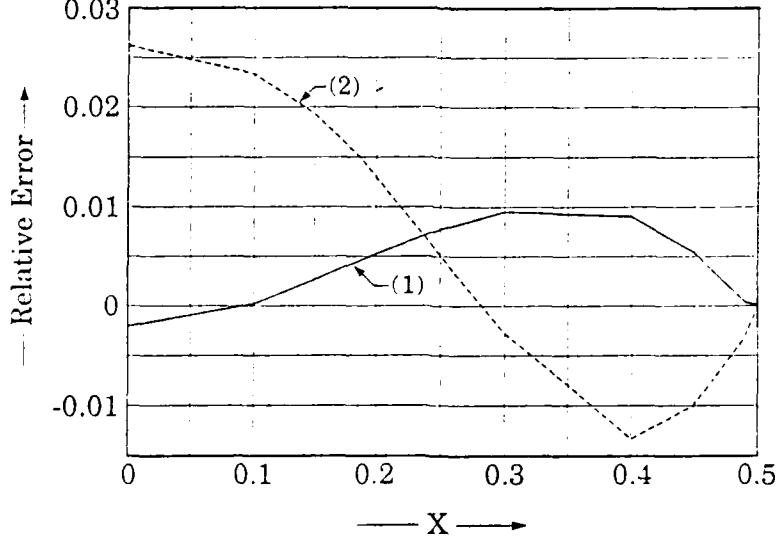


Figure 4: Pointwise error in the average normal deflection for the $(1, 0)$ -model.

- (1) $\kappa = 0.935$ (optimal numerical value)
- (2) $\kappa = \kappa(x)$ as in (5.1)

We see that this x -dependent shear factor performs as well as the best constant value for κ .

So far we addressed the $(1, 0)$ -model. In Figure 5 we show analogous results for the $(1, 2)$ -model. We depict the error in the average normal deflection for the constant shear correction factors $\kappa = 1.0$, $\kappa = 0.874$ from (4.31) and for the variable κ

$$\kappa(x) = 0.874 + (0.833 - 0.874)e^{-70(x-0.5)^2} \quad (5.2)$$

with the value $\kappa = 0.833$ for the nonsmooth solution (see Figure 1). We see once more the same effect as for the $(1, 0)$ -model.

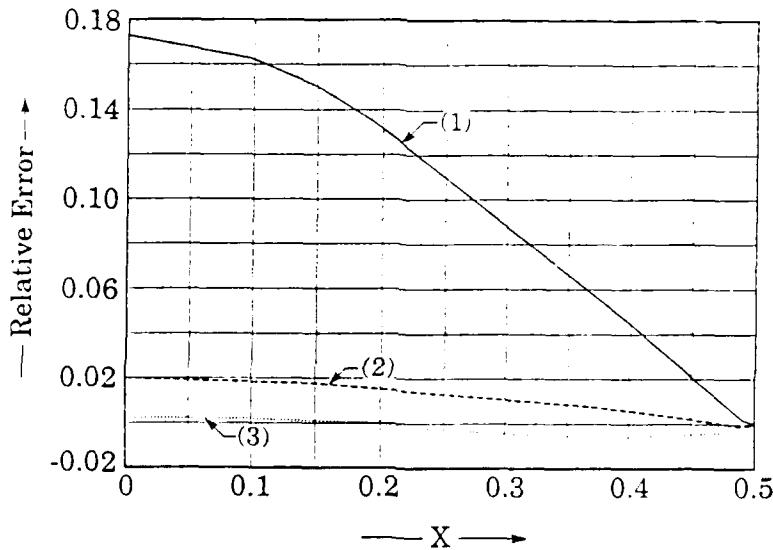


Figure 5: Pointwise error in the average normal deflection for the $(1, 2)$ -model.

- (1) $\kappa = 1$
- (2) $\kappa = 0.874$ (optimal value (4.31))
- (3) $\kappa = \kappa(x)$ (variable correction factor as in (5.2))

Remark 5.1 In practice we would select the shear factor piecewise constant over the finite elements used to solve the beam/ plate problem approximately rather than a function which depends smoothly on x .

Finally we show in Figure 6 that the value of κ_{opt} given by (4.31) which is the same for all criteria used is also in practice (i.e. for the bounded domain) very good for various accuracy measures.

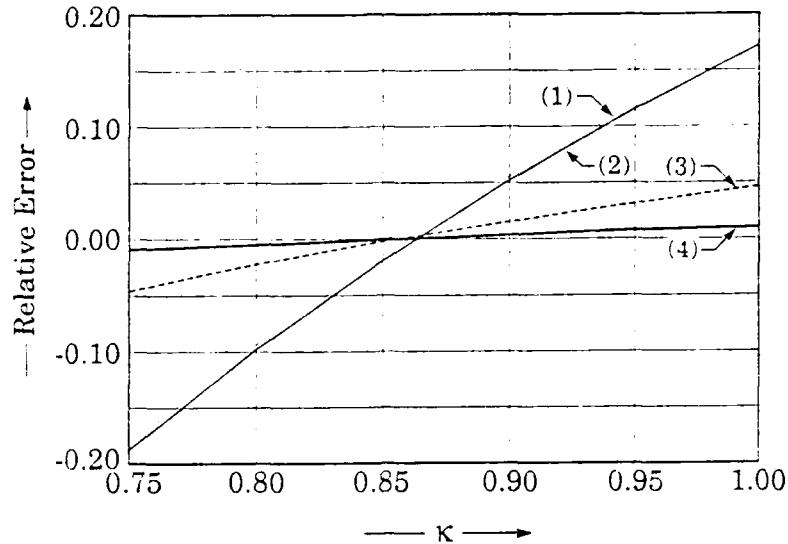


Figure 6: Error in various quantities in dependence on κ for the (1, 2) model.

- (1) $u_3^{(1,2)}(0,0) - u_3(0,0)$
- (2) Error in the average normal deflection at $x_1 = 0$
- (3) $u_3^{(1,2)}(0,0.4) - u_3(0,0.4)$
- (4) Energy of the (1, 2) model - energy of the exact solution

6. Conclusion

We have presented the concept of hierarchical modelling as a numerical method for the approximation of a three dimensional elasticity problem on a “thin” domain. This method can be easily implemented in the context of the h - p version of the finite element method (which is available in the code MSC/PROBE). The selection of the shear correction factors was then based on an optimization of this method. The approach that we have presented suggests concrete values for the shear correction factors, and we found that the optimal choice depends on the aims of computation. We point out that the optimal shear correction factors for the error measures considered here are identical for the (1, 1, 2) model. In contrast, for the Reissner Mindlin (1, 1, 0) model, the optimal shear correction factors strongly depend on the error measures used. We further showed that the optimal shear correction factor should also depend on the smoothness of the solution and consequently have different values in the interior and in a vicinity of the boundary of the plate.

In this context we also remark that the Reissner Mindlin and the (1, 1, 2) model have quite a different boundary layer behaviour on bounded domains [4], [5]. It is possible to use different models from the hierarchy simultaneously in different parts of the plate,

for example, in a neighborhood of the boundary the (1, 1, 2) or a higher order model, while in the interior the (1, 1, 0) model with the proper shear correction factor could be used.

Although we presented here only an analysis of the isotropic case, we emphasize that our approach can also be utilized for anisotropic and laminated materials (where the approximation (2.20) will not be polynomial in x_3 any more) as well as for the analysis of problems other than the classical plate bending problem, and these generalizations will be the subject of a forthcoming paper.

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